УДК 538.9

V. Ya. Krivnov, D. V. Dmitriev

LOW-TEMPERATURE SUSCEPTIBILITY OF THE CLASSICAL FRUSTRATED FERROMAGNETIC SPIN CHAIN

Abstract.

Background. Low-dimensional magnets with competing (frustrated) interactions have attracted much attention last years because these systems have many unusual magnetic properties which are important for application. One of the interest class of such compounds is cuprates consisting of edge-sharing chains with CuO_4 plaquets with the ferromagnetic (F) and the antiferromagnetic (AF) exchange interactions between Cu^{2+} magnetic ions. Of particular interest are the cuprates for which the frustration parameter is close to the critical value corresponding to the quantum phase transition. A minimal model describing the magnetic properties of these cuprates is so-called F-AF chain with the F interaction of nearest-neighbor spins and the AF interaction of next-nearest-neighbor ones. The aim of this work is to study low-temperature magnetic properties of the F-AF chain with the frustration parameter which is close to the critical value. We focus our attention on the behavior of the magnetic susceptibility and the correlation functions in this point.

Materials and methods. The study of the low-temperature thermodynamics of the quantum F-AF chain is a complicated problem. However, there are reasons to expect that the behavior of the low-temperature magnetic properties is universal for the quantum and the classical F-AF chains. Therefore, we consider the classical F-AF chain for which the exact analytical calculations can be provided. They are based on the transfer matrix method adapted to the systems with the competing interactions.

Results. The partition function of the classical F-AF chain with the critical frustration parameter in the low-temperature limit can be reduced to the Schrodinger equation for the quantum particle in the special potential. The eigenvalues and the eigenfunctions of this equation are found. As a result we obtain exact low-temperature asymptotic of the pair correlation functions and the magnetic susceptibility. In the $T \to 0$ limit the susceptibility diverges as $T^{-4/3}$ and the correlation length as $T^{-1/3}$. So, the critical indices of the susceptibility and correlation length are 4/3 and 1/3 correspondingly.

Conclusions. The obtained results demonstrate strong influence of the frustration effects on the magnetic properties of the F-AF chain. In particular, the critical index of the susceptibility is changed from 2 to 4/3 when the frustration parameter is changed from zero to the critical value and from 1 to 1.3 for the correlation length. It is noted that the low-temperature asymptotic with the same critical indices have been obtained for the quantum F-AF chain by spin-wave method. This testifies the universality of the behavior of the magnetic properties of both quantum and the classical F-AF chains.

Key words: frustrated ferromagnetic chain, low-temperature susceptibility, low-dimensional magnet, frustrated interactions, cuprates.

В. Я. Кривнов, Д. В. Дмитриев

НИЗКОТЕМПЕРАТУРНАЯ ВОСПРИИМЧИВОСТЬ КЛАССИЧЕСКОЙ ФРУСТРИРОВАННОЙ ФЕРРОМАГНИТНОЙ ЦЕПОЧКИ

Аннотация.

Актуальность и цели. Низкоразмерные магнетики с конкурирующими (фрустрирующими) взаимодействиями обладают рядом необычных магнитных свойств, важных для практических приложений, и интенсивно исследуются в последние годы. Одними из наиболее интересных систем этого типа являются меднооксидные соединения (купраты), состоящие из цепочек CuO_4 со сравнимыми по величине обменными взаимодействиями ферромагнитного (F) и антиферромагнитного (AF) типов между магнитными ионами Cu^{2+}. Особый интерес представляют купраты, для которых параметр фрустрации (отношение величин AF и F взаимодействий) близок к критическому значению, соответствующему точке квантового фазового перехода. Простейшей моделью, описывающей магнитные свойства этих купратов, является спиновая цепочка с F взаимодействием соседних спинов и AF взаимодействием несоседних (F-AF цепочка). Целью данной работы является изучение низкотемпературных магнитных свойств F-AF цепочки с параметром фрустрации, близким к критическому значению и, в частности, нахождению температурной зависимости магнитной восприимчивости.

Материалы и методы. Исследование низкотемпературной термодинамики квантовой F-AF цепочки представляет весьма сложную в математическом отношении задачу. Вместе с тем есть основания ожидать, что поведение низкотемпературных магнитных свойств является универсальным для квантовой и классической F-AF цепочек. Поэтому в данной работе рассмотрена классическая F-AF цепочка, для которой удается провести точные аналитические вычисления низкотемпературной термодинамики. Они основаны на использовании метода трансфер-матрицы, специально адаптированного к исследованию систем с конкурирующими взаимодействиями.

Результаты. В пределе низких температур статистическая сумма классической F-AF цепочки с параметром фрустрации, близким к критическому, сведена к решению уравнения Шредингера для квантовой частицы в потенциале специального вида. В результате решения соответствующего дифференциального уравнения были найдены собственные значения и собственные функции. Это позволило получить точные низкотемпературные асимптотики парной корреляционной функции и магнитной восприимчивости F-AF цепочки. В пределе $T \to 0$ восприимчивость χ расходится как $\chi \sim T^{-4/3}$, т.е. критический индекс восприимчивости равен 4/3, а корреляционная длина $\sim T^{-1/3}$.

Выводы. Полученные результаты указывает на сильное влияние эффектов фрустрации на магнитные свойства F-AF цепочки, в особенности вблизи точки фазового перехода. В частности, критический индекс восприимчивости изменяется от 2 до 4/3 при изменении параметра фрустрации от нуля до критического значения, а индекс корреляционной длины от 1 до 1/3. Следует отметить, что низкотемпературные асимптотики магнитной восприимчивости и корреляционной длины с такими же критическими индексами были получены в рамках приближенного спин-волнового метода и для квантовой модели. Это свидетельствует об универсальности в поведении магнитных свойств квантовой и классической F-AF цепочек.

Ключевые слова: фрустрирующая ферромагнитная цепочка, низкотемпературная восприимчивость, низкоразмерные магнетики, фрустрирующие взаимодействия, купраты.

Introduction

Strongly frustrated low-dimensional magnets have attracted much attention last years [1]. A very interesting class of such compounds is edge-sharing chains where CuO_4 plaquets are coupled by their edges [2–7]. An important feature of the edge-sharing chains is that the nearest-neighbor (NN) interaction J_1 between Cu spins is ferromagnetic while the next-nearest-neighbor (NNN) interaction J_2 is antiferromagnetic. The competition between them leads to the frustration. A minimal model describing the magnetic properties of these cuprates is so-called F-AF spin chain model the Hamiltonian of which has the form

$$H = J_1 \sum \mathbf{S}_n \mathbf{S}_{n+1} + J_2 \sum \mathbf{S}_n \mathbf{S}_{n+2} \qquad (1)$$

where $\mathbf{S}_n$ is the spin operator on n-th site, and the exchange integrals are $J_1 < 0$ and $J_2 > 0$.

This model is characterized by a frustration parameter $\alpha = J_2 / |J_1|$. The ground state properties of the quantum $s = 1/2$ F-AF chain have been intensively studied last years [8–17]. It is known that the ground state of the model is ferromagnetic for $\alpha < 1/4$. At $\alpha = 1/4$ the ground state phase transition to the incommensurate singlet phase with helical spin correlations takes place. Remarkably, this transition point does not depend on a spin value, including the classical limit $s = \infty$.

However, the influence of the frustration on low-temperature thermodynamics is less studied especially in the vicinity of the ferromagnetic-helimagnet transition point. It is of a particular importance to study this problem, because edge-sharing cuprates with $\alpha \simeq 1/4$ (for example, Li_2CuZrO_4, $Rb_2Cu_2Mo_3O_{12}$) are of special interest [18]. Unfortunately, at present the low-temperature thermodynamics of quantum $s = 1/2$ model (1) at $\alpha \neq 0$ can be studied only either by using of numerical calculations of finite chains or by approximate methods. On the other hand, the classical version of model (1) can be studied by analytical methods giving exact results at $T \to 0$. Of course, the question arises about the relation of these results (in particular, for the susceptibility) to those of the quantum model. It is known [19–21] that the quantum and classical ferromagnetic chains ($\alpha = 0$) have universal low-temperature behavior. Similar universality holds for the dimerized ferromagnetic chains too [22]. As was noted in [20] the physical reason of this universality is the consequence of the fact that the correlation length at $T \to 0$ is larger than de Broglie wavelength of the spin waves. This property is inherent in the frustrated ferromagnetic too. Though such universality for the frustrated ferromagnetic chains is not strictly checked at present, one can expect that the universality holds on for the F-AF chain as well. Therefore, the study of the classical model (1) can be useful for the understanding of the low-temperature properties of the quantum F-AF chains.

At zero temperature classical model (1) has long range-order (LRO) for all values of α: the ferromagnetic LRO at $\alpha \leq 1/4$ and the helical one at $\alpha > 1/4$. At finite temperature the LRO is destroyed by thermal fluctuations and thermodynamic quantities have singular behavior at $T \to 0$. In particular, the zero-field magnetic susceptibility χ diverges. For the 1D Heisenberg ferromagnet (HF) $\chi = 2|J_1|/3T^2$ [23]. At $0 < \alpha < 1/4$ the susceptibility is $\chi = 2(1-4\alpha)|J_1|/3T^2$. This behavior of χ is similar to that for the quantum $s = 1/2$ F-AF model [24].

The value χT^2 vanishes at the transition point indicating the change of the critical exponent. In this paper we focus on the low-temperature behavior of the classical F-AF chains at the ferromagnet-helimagnet transition point, i.e. at $\alpha = 1/4$. This problem is interesting on its own account, because the spectrum of low-energy excitations is proportional to k^4 rather than k^2 as for the HF model. It means that the critical exponents characterizing low-temperature behavior of thermodynamic quantities at $\alpha = 1/4$ can be different from those for the HF chain.

Interesting question is the influence of the frustration on the low-thermodynamics of the model especially near the transition point $\alpha = 1/4$. We study this problem for the classical version of the model (1). At zero temperature the classical model has along range-order (LRO) for all values of α: the ferromagnetic LRO at $\alpha < 1/4$ and the helical one at $\alpha \geq 1/4$. At finite temperature the LRO is destroyed by thermal fluctuations and thermodynamic quantities have a singular behavior at $T \to 0$. In particular, the zero-field susceptibility χ diverges. For the 1D Heisenberg ferromagnet ($\alpha = 0$) $\chi = 2|J_1|/3T^2$ [25]. At $0 < \alpha < 1/4$ the susceptibility is $\chi = 2(1-4\alpha)|J_1|/3T^2$. This behavior of χ is similar to that for the quantum $s = 1/2$ F-AF model [24]. We focus our attention on the behavior of χ in the transition point.

1. Partition function

The partition function Z of the model (1) at $\alpha = 1/4$ is

$$Z = \prod_{n=1}^{N} \int d\Omega_n \exp\left(\sum (\mathbf{S}_n \mathbf{S}_{n+1} - \frac{1}{4} \mathbf{S}_n \mathbf{S}_{n+2}) / t \right) \quad (2)$$

where $\mathbf{S}_n$ is unit vector, $d\Omega_n$ is the volume element of the solid angle for n-th site, $t = T/|J_1|$ and the periodic boundary conditions are proposed.

Our further calculations are based on the transfer matrix method and we use a version of this method adapted to the model with NNN interactions by Harada and Mikeska in [26].

Following to [26] we represent Z in a form

$$Z = \prod_{n=1}^{N} \int d\Omega_n K(\theta_{n-1}, \theta_n; \phi_n), \quad (3)$$

where

$$K(\theta_{n-1}, \theta_n; \phi_n) =$$

$$= \exp\left(\frac{\cos\theta_{n-1} + \cos\theta_n}{2t} - \frac{(\cos\theta_{n-1}\cos\theta_n + \sin\theta_{n-1}\sin\theta_n\cos\phi_n)}{4t} \right), \quad (4)$$

where θ_n is the angle between $\mathbf{S}_n$ and $\mathbf{S}_{n+1}$ and ϕ_n is the angle between components of $\mathbf{S}_{n-1}$ and of $\mathbf{S}_{n+1}$ projected onto (X_n, Y_n) plane of the n-th local coordinate system with the Z_n axis parallel to $\mathbf{S}_n$.

Integrating (4) over ϕ_n we obtain Z in a form

$$Z = \prod_n \int_0^{\pi} d\theta_n \sin\theta_n A(\theta_{n-1}, \theta_n), \tag{5}$$

where

$$A(\theta_{n-1}, \theta_n) = \frac{1}{2} I_0(-z) \exp\left(\frac{\cos\theta_{n-1} + \cos\theta_n}{2t} - \frac{\cos\theta_{n-1}\cos\theta_n}{4t} - \frac{3}{4t} \right), \tag{6}$$

$$z = \frac{\sin\theta_{n-1}\sin\theta_n}{4t}. \tag{7}$$

Let us consider an integral equation

$$\int_0^{\pi} A(\theta_1, \theta_2)\psi_\alpha(\theta_2)\sin(\theta_2)\, d\theta_2 = \lambda_\alpha \psi_\alpha(\theta), \tag{8}$$

where $\psi_\alpha(\theta)$ satisfy normalization condition

$$\int_0^{\pi} \psi_\alpha(\theta)\psi_\beta(\theta)\sin\theta \mathrm{d}\theta = \delta_{\alpha,\beta}. \tag{9}$$

Eigenfunctions ψ_α and eigenvalues λ_α can be chosen as real since the kernel $A(\theta_1, \theta_2)$ is real and symmetric. Then,

$$A(\theta, \theta') = \sum_\alpha \lambda_\alpha \psi_\alpha(\theta)\psi_\alpha(\theta'). \tag{10}$$

Substituting (10) in (5) we obtain in the thermodynamic limit

$$Z = \lambda_0^N, \tag{11}$$

where λ_0 is the largest eigenvalue of (8).

In the low-temperature angles θ_n are small and we can use the asymptotic expansion of the modified Bessel function

$$I_0(-z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + \frac{1}{8z} + O(z^{-2})\right). \tag{12}$$

Then, we expand the expression in the exponent of the transfer matrix to the fourth order in θ_i to obtain

$$A(\theta_1, \theta_2) = \sqrt{\frac{t}{2\pi\theta_1\theta_2}}\left(1 + \frac{t}{2\theta_1\theta_2}\right)\exp\left(-\frac{(\theta_1-\theta_2)^2}{8t} - \frac{\theta_1^2\theta_2^2}{8t} + \frac{(\theta_1-\theta_2)^4}{96t}\right). \tag{13}$$

We can neglect the term $(\theta_1-\theta_2)^4/96t$ as will be seen below.

As a result the integral equation (8) reduces to

$$\int_0^{\pi} \sqrt{\frac{t\theta_2}{2\pi\theta_1}}\left(1 + \frac{t}{2\theta_1\theta_2}\right) e^{-\frac{(\theta_1-\theta_2)^2}{8t} - \frac{\theta_1^2\theta_2^2}{8t}} \psi_\alpha(\theta_2)\, d\theta_2 = \lambda_\alpha \psi_\alpha(\theta_1). \tag{14}$$

The maximum of the expression in the exponent (saddle point) is at $\theta_2 = \theta_1$ (more exactly $\theta_2 = \theta_1 - \theta_1^3 + \ldots$). Near this saddle point we expand $\psi_\alpha(\theta_2)$ as follows

$$\psi_\alpha(\theta_2) = \psi_\alpha(\theta_1) + (\theta_2 - \theta_1)\psi'_\alpha(\theta_1) + \frac{(\theta_2 - \theta_1)^2}{2}\psi''_\alpha(\theta_1) + \ldots \tag{15}$$

and

$$\sqrt{\frac{\theta_2}{\theta_1}} = \sqrt{1 + \frac{\theta_2 - \theta_1}{\theta_1}} = 1 + \frac{\theta_2 - \theta_1}{2\theta_1} - \frac{(\theta_2 - \theta_1)^2}{8\theta_1^2} + \ldots \tag{16}$$

Let us introduce new scaled variables

$$\theta_2 - \theta_1 = t^{1/2}x,$$

$$\theta_1 = t^{1/3}r. \tag{17}$$

Now $\psi_\alpha(\theta) \to \psi_\alpha(r)$ and

$$\psi_\alpha(\theta_2) \to \psi_\alpha(r) + t^{1/6}x\psi'_\alpha(r) + \frac{t^{1/3}x^2}{2}\psi''_\alpha(r) + \ldots \tag{18}$$

$$\sqrt{\frac{\theta_2}{\theta_1}} \to 1 + \frac{t^{1/6}x}{2r} - \frac{t^{1/3}x^2}{8r^2} + O(t^{1/2}), \tag{19}$$

$$\exp\left(-\frac{(\theta_1 - \theta_2)^2}{8t} - \frac{\theta_1^2\theta_2^2}{8t}\right) \to \exp\left(-\frac{x^2}{8} - \frac{t^{1/3}r^4}{8}\right). \tag{20}$$

Summarizing all above we arrive at

$$\int_{-r/t^{1/6}}^{\pi/\sqrt{t}} \left(1 + \frac{t^{1/6}x}{2r} - \frac{t^{1/3}x^2}{8r^2}\right)\left(1 + \frac{t^{1/3}}{2r^2}\right)\left(\psi_\alpha(r) + t^{1/6}x\psi'_\alpha(r) + \frac{t^{1/3}x^2}{2}\psi''_\alpha(r)\right) \times$$

$$\times \exp\left(-\frac{x^2 - t^{1/3}r^4}{8}\right)\frac{tdx}{\sqrt{2\pi}} = \lambda_\alpha\psi_\alpha(r). \tag{21}$$

At $t \to 0$, we can change the limits in the integral to $[-\infty, \infty]$, then only even powers in x gives contribution, so taking into account only terms up to $t^{1/3}$ we obtain

$$\int_{-\infty}^{\infty} \left[\left(1 - \frac{t^{1/3}r^4}{8} + \frac{t^{1/3}}{2r^2}\right)\psi_\alpha + \frac{t^{1/3}x^2}{2}\left(\psi''_\alpha + \frac{1}{r}\psi'_\alpha(r) - \frac{1}{4r^2}\psi_\alpha\right)\right] \times$$

$$\times e^{-x^2/8}\frac{tdx}{\sqrt{2\pi}} = \lambda_\alpha\psi_\alpha. \tag{22}$$

After integration we obtain

$$2t\left(1-\frac{t^{1/3}r^4}{8}+\frac{t^{1/3}}{2r^2}\right)\psi_\alpha+4t^{4/3}\left(\psi_\alpha''+\frac{1}{r}\psi_\alpha'(r)-\frac{1}{4r^2}\psi_\alpha\right)=\lambda_\alpha\psi_\alpha \qquad (23)$$

and, finally,

$$-\psi_\alpha''-\frac{1}{r}\psi_\alpha'+\frac{r^4}{16}\psi_\alpha=\varepsilon_\alpha\psi_\alpha, \qquad (24)$$

with

$$\varepsilon_\alpha=\frac{2t-\lambda_\alpha}{4t^{4/3}}. \qquad (25)$$

So, we have Schrodinger equation for a particle with momentum $l_z=0$ in 2D potential well $U(r)=r^4/16$. The boundary conditions are:

$$\psi_\alpha'(0)=0,\ \psi_\alpha(\infty)=0. \qquad (26)$$

Normalization condition for $\psi_\alpha(r)$ is

$$t^{2/3}\int_0^\infty \psi_\alpha(r)\psi_\beta(r)rdr=\delta_{\alpha,\beta}. \qquad (27)$$

Lowest eigenvalues of equation (24) (corresponding to the largest λ) are

$$\varepsilon_\alpha=0.9305;\ 3.7819;\ 7.435;\ 11.628\ldots \qquad (28)$$

2. Two-spin correlation function

As it was shown in [26] the two-spin correlation function is expressed by following integral

$$\langle \mathbf{S}_1\cdot\mathbf{S}_{1+n}\rangle=\frac{1}{\lambda_0^{n-1}}\int_0^\pi d\theta_n\sin\theta_n\times$$

$$\times\prod_{l=1}^{n-1}d\theta_l\sin\theta_l\psi_0(\theta_l)\psi_0(\theta_n)\begin{pmatrix}0 & 1\end{pmatrix}B(\theta_1)H(\theta_l,\theta_{l+1})B(\theta_n)\begin{pmatrix}0\\1\end{pmatrix}, \qquad (29)$$

where

$$B(\theta)=\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}, \qquad (30)$$

$$H(\theta,\theta')=B(\theta)\begin{pmatrix}-\tilde{A}(\theta,\theta') & 0\\ 0 & A(\theta,\theta')\end{pmatrix}B(\theta'), \qquad (31)$$

and $\tilde{A}(\theta,\theta')$ is given by (6) with $I_0(-z)$ replaced by $I_1(-z)$.

Using the asymptotic expansion of the Bessel function

$$I_1(-z) = -\frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{3}{8z} + O(z^{-2})\right), \tag{32}$$

we obtain

$$H(\theta,\theta') = A_0(\theta,\theta')\begin{pmatrix} 1 - \frac{3t}{2\theta\theta'} & \frac{\theta+\theta'}{2} \\ -\frac{\theta+\theta'}{2} & 1 + \frac{t}{2\theta\theta'} \end{pmatrix}, \tag{33}$$

where

$$A_0(\theta,\theta') = \sqrt{\frac{t}{2\pi\theta\theta'}}\exp\left(-\frac{(\theta-\theta')^2}{8t} - \frac{\theta^2\theta'^2}{8t}\right). \tag{34}$$

The matrix $H(\theta,\theta')$ is not symmetric. Therefore, to calculate $\langle \mathbf{S}_1 \cdot \mathbf{S}_{1+n} \rangle$ it is necessary to solve a pair of the integral equations

$$\int_0^{\pi} H(\theta_1,\theta_2)\vec{u}_\alpha(\theta_2)\sin(\theta_2)\,d\theta_2 = \eta_\alpha \vec{u}_\alpha(\theta_1), \tag{35}$$

$$\int_0^{\pi} H^T(\theta_1,\theta_2)\vec{v}_\alpha(\theta_2)\sin(\theta_2)\,d\theta_2 = \eta_\alpha \vec{v}_\alpha(\theta_1), \tag{36}$$

where $H^T(\theta_1,\theta_2)$ is transposed matrix $H(\theta_1,\theta_2)$ and two-component vectors $\vec{u}_\alpha$ and $\vec{v}_\alpha$

$$\vec{u}_\alpha = \begin{pmatrix} u_{1,\alpha} \\ u_{2,\alpha} \end{pmatrix}, \qquad \vec{v}_\alpha = \begin{pmatrix} v_{1,\alpha} \\ v_{2,\alpha} \end{pmatrix} \tag{37}$$

satisfy orthonormality relations,

$$\int_0^{\pi} \vec{u}_\alpha^T(\theta)\vec{v}_\beta(\theta)\sin(\theta)\,d\theta = \int_0^{\pi} \vec{v}_\alpha^T(\theta)\vec{u}_\beta(\theta)\sin(\theta)\,d\theta = \delta_{\alpha,\beta} \tag{38}$$

Then,

$$H(\theta,\theta') = \sum_\alpha \eta_\alpha \vec{u}_\alpha(\theta)\vec{v}_\alpha^T(\theta'). \tag{39}$$

At small θ_1, θ_2 equations (35) and (36) reduce to

$$\int_0^{\pi} A_0(\theta_1,\theta_2)\left[\left(1 - \frac{3t}{2\theta_1\theta_2}\right)u_{1,\alpha}(\theta_2) + \theta_1 u_{2,\alpha}(\theta_2)\right]\sin(\theta_2)\,d\theta_2 = \eta_\alpha u_{1,\alpha}(\theta_1), \tag{40}$$

$$\int_0^{\pi} A_0(\theta_1,\theta_2)\left[-\theta_1 u_{1,\alpha}(\theta_2) + \left(1 + \frac{t}{2\theta_1\theta_2}\right)u_{2,\alpha}(\theta_2)\right]\sin(\theta_2)\,d\theta_2 = \eta_\alpha u_{2,\alpha}(\theta_1). \tag{41}$$

Integrating these equations near the saddle point similar to done above, we get a pair of linear differential equation

$$2t\left(1-\frac{t^{1/3}r^4}{8}-\frac{3t^{1/3}}{2r^2}\right)u_{1,\alpha}+4t^{4/3}\left(u''_{1,\alpha}+\frac{1}{r}u'_{1,\alpha}-\frac{1}{4r^2}u_{1,\alpha}\right)+$$

$$+2t^{4/3}ru_{2,\alpha}=\eta_\alpha u_{1,\alpha}, \tag{42}$$

$$2t\left(1-\frac{t^{1/3}r^4}{8}+\frac{t^{1/3}}{2r^2}\right)u_{2,\alpha}+4t^{4/3}\left(u''_{2,\alpha}+\frac{1}{r}u'_{2,\alpha}-\frac{1}{4r^2}u_{2,\alpha}\right)-$$

$$-2t^{4/3}ru_{1,\alpha}=\eta_\alpha u_{2,\alpha}, \tag{43}$$

and, finally,

$$-u''_{1,\alpha}-\frac{1}{r}u'_{1,\alpha}+\frac{1}{r^2}u_{1,\alpha}+\frac{r^4}{16}u_{1,\alpha}+\frac{r}{2}u_{2,\alpha}=\mu_\alpha u_{1,\alpha} \tag{44}$$

$$-u''_{2,\alpha}-\frac{1}{r}u'_{2,\alpha}+\frac{r^4}{16}u_{2,\alpha}-\frac{r}{2}u_{1,\alpha}=\mu_\alpha u_{2,\alpha}, \tag{45}$$

where

$$\mu_\alpha=\frac{2t-\eta_\alpha}{4t^{4/3}}. \tag{46}$$

The lowest eigenvalues of equations (44) and (45) are

$$\mu_\alpha=1.4113;\ 1.8294;\ 3.983;\ 5.357;\ 7.576\ldots \tag{47}$$

For $\vec{v}_\alpha$ similar procedure gives

$$-v''_{1,\alpha}-\frac{1}{r}v'_{1,\alpha}+\frac{1}{r^2}v_{1,\alpha}+\frac{r^4}{16}v_{1,\alpha}-\frac{r}{2}v_{2,\alpha}=\mu_\alpha v_{1,\alpha}, \tag{48}$$

$$-v''_{2,\alpha}-\frac{1}{r}v'_{2,\alpha}+\frac{r^4}{16}v_{2,\alpha}+\frac{r}{2}v_{1,\alpha}=\mu_\alpha v_{2,\alpha}. \tag{49}$$

It follows from Eqs. (44)–(45) and (48)–(49) the function $\vec{v}_\alpha$ is connected with $\vec{u}_\alpha$ by relations $v_{1,\alpha}=-u_{1,\alpha}$, $v_{2,\alpha}=u_{2,\alpha}$ and therefore

$$t^{2/3}\int_0^\infty\left(u_{2,\alpha}u_{2,\beta}-u_{1,\alpha}u_{1,\beta}\right)rdr=\delta_{\alpha,\beta}. \tag{50}$$

Using (39) and (38) we obtain the correlation function (29) in a form

$$\langle \mathbf{S}_1\cdot\mathbf{S}_{1+n}\rangle=\sum_\alpha y_\alpha^{n-1}f_\alpha^2, \tag{51}$$

where $y_\alpha=\eta_\alpha/\lambda_0$ and

$$f_\alpha = \int_0^\pi \psi_0(\theta) u_{2,\alpha}(\theta) \sin(\theta) d\theta = t^{2/3} \int_0^\infty \psi_0(r) u_{2,\alpha}(r) r dr . \tag{52}$$

At $t \to 0$

$$y_\alpha = \frac{2t - 4t^{4/3}\mu_\alpha}{2t - 4t^{4/3}\varepsilon_0} \approx 1 - 2t^{1/3}(\mu_\alpha - \varepsilon_0) \tag{53}$$

and the correlation function is

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_{1+n} \rangle = \sum_\alpha f_\alpha^2 \exp\left\{-(n-1)2t^{1/3}((\mu_\alpha - \varepsilon_0)\right\}. \tag{54}$$

According to (54) the correlation length ξ is

$$\xi = \frac{1}{2(\mu_1 - \varepsilon_0)t^{1/3}} = \frac{1.04}{t^{1/3}}. \tag{55}$$

Then susceptibility at $T \to 0$ is

$$\chi = \frac{1}{3TN} \sum_n \langle \mathbf{S}_1 \cdot \mathbf{S}_{1+n} \rangle = \frac{1}{3T}\left(1 + 2\sum_\alpha \frac{f_\alpha^2}{1 - y_\alpha}\right) = \frac{1}{3T}\left(1 + t^{-1/3} \sum_\alpha \frac{f_\alpha^2}{\mu_\alpha - \varepsilon_0}\right). \tag{56}$$

Using normalization conditions we can rewrite

$$f_\alpha^2 = \frac{\left[\int_0^\infty \psi_0(r) u_{2,\alpha}(r) r dr\right]^2}{\int_0^\infty \psi_0^2(r) r dr \int_0^\infty \left(u_{2,\alpha}^2 - u_{1,\alpha}^2\right) r dr} \tag{57}$$

which is independent of normalization and, therefore, is more convenient for numerical calculations.

Now we see that f_α^2 and $(\mu_\alpha - \varepsilon_0)$ depends on solution of differential equations independent of t. So, the sum in χ gives numerical constant

$$\sum_\alpha \frac{f_\alpha^2}{\mu_\alpha - \varepsilon_0} = 3C. \tag{58}$$

Therefore, the low-temperature susceptibility behaves as

$$\chi = \frac{C\left|J_1^{1/3}\right|}{T^{4/3}}. \tag{59}$$

Numerical calculations gives for the constant C the value $C = 1.070(1)$.

Thus, the critical index for the susceptibility in the critical point is 4/3 and that for the correlation length is 1/3.

Conclusions

We have obtained the exact results for the low-temperature thermodynamics of the classical F-AF model at the frustration parameter $\alpha = 1/4$, where the ground state phase transition from the ferromagnetic to the helical phase occurs. The main result relates to the behavior of the zero-field susceptibility χ and the correlation length l_c. It is shown that the critical exponents of χ and l_c are changed from 2 to $4/3$ and from 1 to $1/3$ correspondingly, when $\alpha \to 1/4$ from the ferromagnetic side. In [27, 28] we have considered a continuum version of the model (1). In the continuum limit the calculation of the partition function and the correlation function is reduced to quantum problem of a particle in a potential well. Remarkably, the exact low-temperature asymptotes of χ and l_c of the continuum and lattice model coincide. However, the use of the continuum approximation gives the leading term of the low-temperature asymptotes only. The present method allows to obtain next leading term of the asymptotes of χ. It turns out that it is proportional to $t^{-1/3}$.

It is interesting to compare the exact expression for the susceptibility with the results found by approximate and numerical methods. The numerical data [29] obtained using Monte Carlo simulations confirm the analytical result (59). One of the analytical methods is the modified spin-wave theory (MSWT) proposed by Takahashi [25] to extend the spin-wave theory to the low-dimensional spin systems without LRO. Remarkably, this method gives the true critical exponent $4/3$ for the susceptibility behavior: $\chi = ct^{-4/3}$. However, the numerical coefficient c differs from the exact one and the MSWT result is $c = 1.19$. It is interesting to note that the MSWT gives the exact low-temperature asymptotic of χ for the classical ferromagnetic chain ($\alpha = 0$), where $\chi = (2/3)t^{-2}$ [23]. Moreover, the MSWT gives the exact result for χ at $T \to 0$ for the quantum ferromagnetic chain with $s = 1/2$ as well. As was noted in Introduction, the quantum and the classical ferromagnetic chains have universal low-temperature properties. The low-temperature susceptibility of the ferromagnetic chain is described by the scaling function which is valid for any value of spin s. Though for the F-AF model at $\alpha = 1/4$ there is no rigorous proof of such universality we expect that for the quantum F-AF chain the critical exponent of χ is the same as in the classical model.

References

1. **Mikeska H.-J., Kolezhuk A. K.** *Quantum Magnetism*, Lecture Notes in Physics. 2004, vol. 645, p. 1. Edited by U. Schollwöck, J. Richter, D. J. J. Farnell, and R. F. Bishop, Eds. (Springer-Verlag, Berlin).
2. **Mizuno Y., Tohyama T., Maekawa S., Osafune T., Motoyama N., Eisaki H., Uchida S.** *Phys. Rev. B*. 1998, vol. 57, p. 5326.
3. **Masuda T., Zheludev A., Bush A., Markina M., Vasiliev A.** *Phys. Rev. Lett*. 2004, vol. 92, p. 177201.
4. **Hase M., Kuroe H., Ozawa K., Suzuki O., Kitazawa H., Kido G., Sekine T.** *Phys. Rev. B*. 2004, vol. 70, p. 104426.
5. **Drechsler S.-L., Malek J., Richter J., Moskvin A. S., Gippius A. A., Rosner H.** *Phys. Rev. Lett*. 2005, vol. 94, p. 039705.
6. **Capogna L., Mayr M., Horsch P., Raichle M., Kremer R. K., Sofin M., Malek A., Jansen M., Keimer B.** *Phys. Rev. B*. 2005, vol. 71, p. 140402.
7. **Malek J., Drechsler S. L., Nitzsche U., Rosner H., Eschrig H**. *Phys. Rev. B*. 2008, vol.78, p. 060508(R).
8. **Chubukov A.V.** *Phys. Rev. B*. 1991, vol. 44, p. 4693.
9. **Nersesyan A. A., Gogolin A. O., Essler F. H. L.** *Phys. Rev. Lett*. 1998, vol. 81, p. 910.

10. **Dmitriev D. V., Krivnov V. Ya.** *Phys. Rev. B*. 2006, vol. 73, p. 024402.
11. **Heidrich-Meisner F., Honecker A., Vekua T.** *Phys. Rev. B*. 2006, vol. 74, p. 020403(R).
12. **Hikihara T., Kecke L., Momoi T., Furusaki A.** *Phys. Rev. B*. 2008, vol. 78, p. 144404.
13. **Sudan J., Luscher A., Laeuchli A. M.** *Phys. Rev. B*. 2009, vol. 80, p. 140402.
14. **Itoi C., Qin S.** *Phys. Rev. B*. 2001, vol. 63, p. 224423.
15. **Lu H. T., Wang Y. J., Qin S., Xiang T.** *Phys. Rev. B*. 2006, vol. 74, p. 134425.
16. **Dmitriev D. V., Krivnov V. Ya., Richter J.** *Phys. Rev. B*. 2007, vol. 75, p. 014424.
17. **Kuzian R., Drechsler S.-L.** *Phys. Rev. B*. 2007, vol. 75, p. 024401.
18. **Drechsler S.-L., Volkova O., Vasiliev A. N., Tristan N., Richter J., Schmidt M., Rosner R., Malek J., Klingeler R., Zvyagin A. A., Buechner R.** *Phys. Rev. Lett*. 2007, vol. 98, p. 077202.
19. **Nakamura H., Takahashi M.** *J. Phys. Soc. Jpn*. 1994, vol. 63, p. 2563.
20. **Takahashi M., H. Nakamura H., Sachdev S.** *Phys. Rev. B*. 1996, vol. 54, p. 744.
21. **Theordorakopoulos N., Bacalis N. C.** *Phys. Rev. B*. 1997, vol. 55, p. 52.
22. **Dmitriev D. V., Krivnov V. Ya.** *Phys. Rev. B*. 2012, vol. 86, p. 134407.
23. **Fisher M. E.** *Am. J. Phys*. 1964, vol. 32, p. 343.
24. **Hartel M., Richter J., Ihle D., Drechsler S.-L.** *Phys. Rev. B*. 2008, vol. 78, p. 174412.
25. **Takahashi M.** *Phys. Rev. Lett*. 1987, vol. 58, p. 168.
26. **Harada I., Mikeska H. J.** *Z.Phys.B-Condensed Matter*. 1998, vol. 72, p. 391.
27. **Dmitriev D. V., Krivnov V. Ya**. *Phys. Rev. B*. 2010, vol. 82, p. 054407.
28. **Dmitriev D. V., Krivnov V. Ya., Kuzminyh N. Yu.** *Phys. Rev. B*. 2011, vol. 84, p. 214438.
29. **Sirker J., Krivnov V. Ya., Dmitriev D. V., Herzog A., Janson O., Nishimoto S., Drechsler S.-L., Richter J.** *Phys.Rev.B*. 2011, vol. 84, p. 144403.

Кривнов Валерий Яковлевич
доктор физико-математических наук, профессор, заведующий отделом, Институт биохимической физики имени Н. М. Эмануэля РАН (Россия, г. Москва, ул. Косыгина, 4)

E-mail: krivnov@deom.chph.ras.ru

Krivnov Valeriy Yakovlevich
Doctor of physical and mathematical sciences, professor, head of department, Emanuel Institute of Biochemical Physics, Russian Academy of Sciences (4 Kosygina street, Moscow, Russia)

Дмитриев Дмитрий Владимирович
кандидат физико-математических наук, старший научный сотрудник, Институт биохимической физики имени Н. М. Эмануэля РАН (Россия, г. Москва, ул. Косыгина, 4)

E-mail: krivnov@deom.chph.ras.ru

Dmitriev Dmitriy Vladimirovich
Candidate of physical and mathematical sciences, senior staff scientist, Emanuel Institute of Biochemical Physics, Russian Academy of Sciences (4 Kosygina street, Moscow, Russia)

УДК 538.9

Krivnov, V. Ya.

Low-temperature susceptibility of the classical frustrated ferromagnetic spin chain / V. Ya. Krivnov, D. V. Dmitriev // Известия высших учебных заведений. Поволжский регион. Физико-математические науки. – 2014. – № 4 (32). – С. 96–107.